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# Modification terms to the Black-Scholes model based on the functional volatility model

Xiaozheng Lin*, Wei Chen, Meiqing Wang
College of Mathematics and Computer Science
Fuzhou University
Fuzhou, China
Email: lynn.xiaozheng@gmail.com;965340978@qq.com; mqwang@fzu.edu.cn

Choi-Hong Lai
Department of Mathematics
University of Greenwich
London,SE10 9LS,UK
Email: C.H.Lai@greenwich.ac.uk

***Abstract*—The Black-Scholes model (B-S model) is a common used option pricing model, which assumes that the volatility and the interest rate are constants. However, the empirical analysis shows that the option prices obtained from the B-S model are quite different from the market prices. The traditional improvement to the B-S model is to replace the volatility constants or interest rate constants with variables. This paper proposes the concept of modification term of Black-Scholes model, which is different from the traditional methods.The errors of the B-S model may be calculated through the market data, and is fitted as a function of a functional implied volatility and the option gamma. The fitted function can be considered as the modification term of the Black-Scholes model. The functional implied volatility can be constructed by using the Gaussian semi-parametric implied volatility model proposed by Ref. [4]. Experimental results show that the modified model has a better predictive effect on option pricing.**

***Index Terms*—right-term values;modification term;functional implied volatility;modified Black-Scholes model**

## I. Introduction

In the early 1970s, Fischer Black and Myron Scholes proposed the first complete option pricing model, deriving the famous Black-Scholes option pricing model (also known as the Black-Scholes model) under a series of strict assumptions [1].

The Black-Scholes model requires the assumption that the underlying asset volatilities are constant [1]. It is practically impossible to execute in real markets. Therefore, option pricing results using constant volatilities in the Black-Scholes partial differential equation (PDE) are inevitably to be different from pricing results using implied volatilities [2]. The objective of this paper is to address this problem through the use of functional volatility model instead of the constant historical volatility to generate the modification term of the Black-Scholes model.

The paper is organised as follows. Section II discusses the modification term to the Black-Scholes model when an suitable implied volatility model is used. Section III provides a functional implied volatility model and introduces the modification term based on the functional implied volatility model we established. Section IV gives the corresponding numerical experiments and analysis of the method proposed in the previous. Finally, conclusions are drawn.

## II. Modification Term of The Black-Scholes Model

### A. Black-Scholes Model and Its Modification Term

Under a series of strict assumptions, the relationship between option price, underlying asset price and time can be described by the following famous Black-Scholes partial differential equation(PDE) [1]:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV, \tag{1}$$

subject to suitable terminal and boundary conditions provides deterministic price of many options and derivatives. Here $S$ denote the underlying asset price, $V$ is the option price, $r$ is the risk-free rate and $\sigma$ means the underlying asset volatility. The volatility of the underlying asset calculated inversely from the price of a single option becomes the implied volatility [1]. The Black-Scholes model assumes that the volatility of the underlying asset $\sigma$ is constant.

Perform simple item shift processing on model (1):

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0. \tag{2}$$

It can be found from equation (2) that if the Black-Scholes model has no errors at all, then substituting the actual option price into the left of equation (2) for derivative calculation, then the right-term should be a constant 0 [3].

However, a large number of numerical experimental studies have found that the right term is non-zero, which shows that the B-S model under strict assumptions does not reflect the actual market conditions well and the non-zero right term can be considered as the errors of the B-S model for the actual market [3].

As shown in Fig.1, taking the 2019 SSE 50ETF purchase of December 2950 option as an example, substituting the actual market price data of this option into equation (2) to obtain the logarithm value of the right-term.

In this paper, the right-term values are used as the modification term $f$ of the Black-Scholes model:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = f. \tag{3}$$

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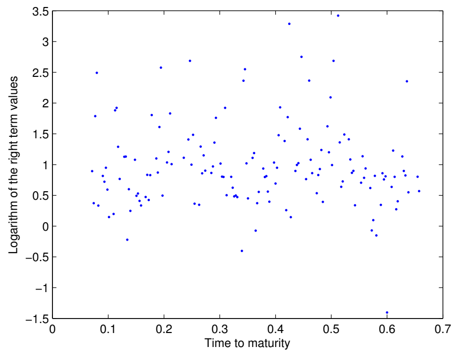



Fig. 1. Logarithm of the right term

Therefore, if we can model the right-term $f$ calculated from the actual transaction data, it is equivalent to finding the modification term function of the B-S model, and the original Black-Scholes model can be modified according to the modification term.

### B. Modification Term Based On Implied Volatility Model

The assumption that the volatility of the underlying asset is a constant is one of the reasons for the error of the traditional Black-Scholes model. Therefore, in order to model the actual right-term $f$ and amend the Black-Scholes model, the most direct solution is to use an implied volatility model that can characterize the time-varying characteristics of underlying asset volatilities to derive the modification term of the Black-Scholes option pricing formula.

Since the implied volatility of the underlying asset price is calculated by substituting the actual option price into the Black-Scholes model, then equation(4) can be obtained when a suitable model (4) is considered to be a good fit for the actual implied volatility. Replace $\sigma$ in equation(3) with the suitable implied volatility model $\widehat{\sigma}$:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\widehat{\sigma}^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0. \tag{4}$$

The difference of equation (3) minus equation (4) is used as the modification term $f$ of Black-Scholes model,

$$f = \frac{1}{2}\left(\sigma^2 - \widehat{\sigma}^2\right) S^2 \frac{\partial^2 V}{\partial S^2}. \tag{5}$$

Therefore, the problem of B-S model modification from the perspective of right-term modelling is transformed into a problem of finding a better implied volatility model $\widehat{\sigma}$ to generate modification term equation(5).

## III. The Modification Term Based on The Functional Implied Volatility Model

### A. The Gaussian semi-parametric implied volatility model

Contrary to the assumption that the implied volatility is constant in the Black-Scholes model, many scholars have found that implied volatility has the structural characteristics of smile, smirk, and skew in relation to execution price.On the other hand,many studies also show the relationship between implied volatility and the maturity date of an option.

In order to better express the degree of curvature and translation position of the implied volatility delivery structure, Ref. [4] used the inverse Gaussian function and parametric exponential variables to improve the implied volatility surface model proposed by Cassese & Guidolin [5],and obtained the Gaussian semi-parametric implied volatility model:

$$\begin{aligned}\widehat{\sigma}_\tau(M) &= x_{1,\tau} + x_{2,\tau}\left(M - x_{3,\tau}\right) \\ &+ x_{4,\tau}[\max(\frac{1}{\theta\sqrt{2\pi}} e^{-\frac{(M-x_{3,r})^2}{2\theta^2}}) - \frac{1}{\theta\sqrt{2\pi}} e^{-\frac{(M-x_{3,r})^2}{2\theta^2}}]^{\alpha_\tau} \\ &+ \varepsilon_\tau,\end{aligned} \tag{6}$$

here $M$ denote the moneyness:

$$M = \ln\left(\frac{S}{K}\right), \tag{7}$$

$x_{1,\tau}$ and $x_{3,\tau}$ are parameters that determine the translational position of the fitted curve,$x_{3,\tau}$ is the mathematical expectation of $M$.$x_{2,\tau}$,$x_{4,\tau}$ and $\varepsilon_\tau$ are parameters that determine the bending amplitude of the fitted curve.$\theta$ is the variance of $M$. They are all fitted and calculated from market data [4].

### B. Functional Implied Volatility Model

Model (6) can well simulate the implied volatility when the remaining period is $\tau$ in a certain trading day $t$ [4]. In other words, model (6) fixes the time to maturity $\tau$, and only considers the influence of the moneyness $M$ within each $\tau$. Functional implied volatility model proposed in this paper provides a model which can consider the influence of both moneyness $M$ and the time to maturity $\tau$:

$$\begin{aligned}\widehat{\sigma}(\tau, M) =& f_1(\tau) + f_2(\tau)(M - \mu_M) + f_3(\tau)[\max(\frac{1}{\theta\sqrt{2\pi}} \\ & e^{-\frac{(M-\mu_M)^2}{2\theta^2}}) - \frac{1}{\theta\sqrt{2\pi}} e^{-\frac{(M-\mu_M)^2}{2\theta^2}}]^{\alpha_\tau} + \varepsilon_\tau,\end{aligned} \tag{8}$$

$\mu_M$ represents $x_{3,\tau}$ in equation (6), which is the mean value of moneyness $M$.

In order to construct the randomness of the implied volatility surface, a fourier function is used to replace the constant $x_{1,\tau}$ in the case of a fixed remaining term $\tau$:

$$f_1(\tau) = a_0 + a_1\cos(w * \tau) + b_1\sin(w * \tau). \tag{9}$$

Consider the randomness and asymmetry in the $\tau$ direction:

$$f_2(\tau) = a_2\sin(b_2\tau + c_2), \tag{10}$$

$$f_3(\tau) = a_3\exp(b\tau), \tag{11}$$

here $f_2(\tau)$ denote the function sum of sine, and $f_3(\tau)$ is a exponential function.Equations (8) − (11) constitute the functional implied volatility model proposed in this paper.

### C. The Modification Term Based on The Functional Implied Volatility Model

The functional implied volatility model $(8)-(11)$ is used as a suitable implied volatility model of equation(4). According to equation(5),the modification term of the Black-Scholes model should be:

$$f=\frac{1}{2}\left(\sigma^{2}-\widehat{\sigma}^{2}(\tau, M)\right) S^{2} \frac{\partial^{2} V}{\partial S^{2}}. \tag{12}$$

Therefore, the Black-Scholes model with modification term function (12) is:

$$\begin{aligned} &\frac{\partial V}{\partial t}+r S \frac{\partial V}{\partial S}+\frac{1}{2} \sigma^{2} S^{2} \frac{\partial^{2} V}{\partial S^{2}}-r V \\ &=\frac{1}{2}\left(\sigma^{2}-\widehat{\sigma}^{2}(\tau, M)\right) S^{2} \frac{\partial^{2} V}{\partial S^{2}}, \end{aligned} \tag{13}$$

$\sigma$ represents the constant volatility of the underlying asset in the original Black-Scholes model (1) and $\widehat{\sigma}(\tau, M)$ are the implied volatilities calculated from the functional implied volatility model(8)-(11) under a certain time to maturity $\tau$ and a moneyness $M$.

Therefore, equations(8)-(13) constitute the modified Black-Scholes model.

## IV. Numerical Experiments and Analysis

### A. Data Preprocessing

This paper examines 9 European-style call options on the SSE 50ETF in December 2019. Under every different remaining terms $t_{ij}, i=1,2,...,9, j=1,2,...,n_i(t_{i1}=0, t_{iN}=T)$, these 9 options had the same maturity date $T$, risk-free interest rate $r$ and the same underlying asset prices $S(t_{ij})$ with different strike prices $K_i$.

The numerical experimental part of this section will show and analyze the fitting results of the functional implied volatility model, the construction accuracy of the modification term and the forecast accuracy of option pricing on the SSE 50ETF in December 2020.

All data are sourced from the Cathay Financial Database.

### B. The Fitting Results of The Functional Implied Volatility Model

The following figure 2 shows the functional implied volatility surface (8)'s fitting results of 9 options on the SSE 50ETF in December 2020.

It can be seen from Fig.2 that the functional implied volatility surface model constructed in this paper can fit the actual implied volatilities well.

### C. The Construction Accuracy of The Modification Term

In the numerical experiment of this paper, the modification term values calculated by equation (12) for 9 options were compared with the actual modification term values $f_{true}$ which were calculated by equation (3) using the actual market data.

From the comparison results, we can observe the accuracy of the modification term equation (12) of the Black-Scholes model proposed in this paper. Consider using the Euclidean

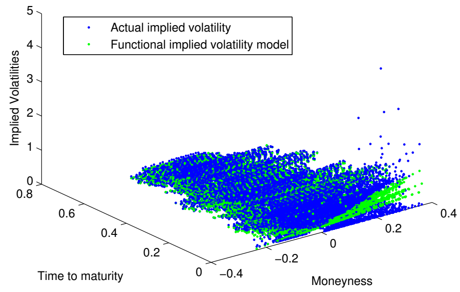



Fig. 2. Fitting results of functional implied volatility models for 9 European call options

distance as a measure of the modification term values obtained by equation (12) and the actual right-term values:

$$d=\sqrt{\sum_{i=1}^{N}\left(f\left(t_{i}\right)-f_{\text{true}}\left(t_{i}\right)\right)^{2}}, \tag{14}$$

this paper selects the minimum, maximum and average Euclidean distances of the 9 options used in the experiment to illustrate the accuracy of the modification term equation (12).

TABLE I
THE CONSTRUCTION ACCURACY OF THE MODIFICATION TERM

| Euclidean distance | minimum | maximum | average |
|---|---|---|---|
| Values | 0.1350 | 2.3980 | 1.2696 |

It can be seen from Table 1 that the fitting degree between the modification term values $f$ obtained by equation (12) and the actual modification term values $f_{true}$ is relatively high.

### D. The Performance of The Modificated Black-Scholes Model on the Accuracy of Option Prices

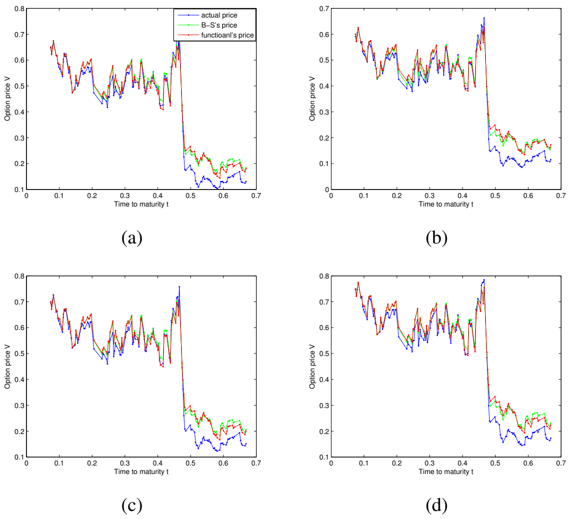
The experiment used the modified Black-Scholes model (8)-(13) to predict the 2020 option prices corresponding to the 9 options used in the experiment, and compared the predicted results with the actual option prices in 2020.

Fig.3 shows the comparison between the results of the modified Black-Scholes model (8)-(13) and the actual option prices in 2020(4 out of 9 options).

It can be seen from Fig.3 that the modified Black-Scholes model results are closer to the actual values than the traditional Black-Scholes model results.

### E. Model Analysis and Evaluation

The main purpose of this paper is to establish the modification term of the Black-Scholes model from the right-end error calculated by substituting the actual market option price into the Black-Scholes equation(3), and at the same time try to use the modified model to predict option prices.




(a) (b) (c) (d)

Fig. 3. Results of the modified Black-Scholes model on the accuracy of option prices

From the numerical experiment results, it can be found that the simulation results of the modification term model (8)-(12) proposed in this paper can fit the right-end errors of the actual model well,but lacking in predicting option prices.This reminds us that we can study the modification term and prediction accuracy of the Black-Scholes model from the aspects of the computational stability and well-posedness of the differential equations (8)-(13).

But even so, this paper provides a novel idea for the modification of the Black-Scholes model, which is innovative.And under this idea, a modification term with a high degree of fit is constructed, so that the follow-up work can be carried out based on this modification term or the modification idea.

## V. Conclusion

This paper proposes the concept of the modification term of the Black-Scholes model, and uses actual market transaction data to calculate the error values at the right-term of the Black-Scholes equation (3), and then establishes the modification term model (8)-(12) of the Black-Scholes model.In this paper, numerical experiments are designed to start with the accuracy of the construction of the modification term model (8)-(12) and the accuracy of using the modified equation (8)-(13) to predict option prices.

The empirical analysis shows that the modification term model (8)-(12) can fit the actual right-end errors of the Black-Scholes equation well. At the same time, the modified Black-Scholes model can also predict option prices more accurately.

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