Multiple solutions for the nonhomogeneous Schrödinger-Poisson equations involving the fractional Laplacian

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***Abstract*—In this work, we consider the Schrödinger-Poisson equations with the fractional Laplacian of order α in R^N. Under some suitable assumptions on potential $V(x)$ and the nonlinear term $f(x,u)$, the existence of multiple solutions is proved by using the Mountain Pass Theorem and the Ekelands variational principle in critical point theory. As a special case, we prove the existence of two nontrivial radial solutions when $V(x) \equiv 1, f(x,u) = |u|^{p-2}u$.**

***Keywords*-Nonhomogeneous; Schrödinger-Poisson equations; Mountain Pass Theorem; Variational methods**

I. INTRODUCTION

In this paper, we are concerned with the following fractional Schrödinger-Poissson equation

$$\begin{cases} (-\Delta)^\alpha u + V(x)u + \phi u = f(x,u) + h(x), & x \in \mathbb{R}^N, \\ (-\Delta)^\alpha \phi = K_\alpha u^2, & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $\alpha \in (0,1)$, $K_\alpha = \frac{\pi^{-\alpha}\Gamma(\alpha)}{\pi^{-(3-2\alpha)/2}\Gamma((3-2\alpha)/2)}$, $(-\Delta)^\alpha$ stands for the fractional Laplacian, $u,\ \phi : \mathbb{R}^N \to \mathbb{R}$,$f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$.

$$\mathcal{F}((-\Delta)^\alpha \omega)(\xi) = |\xi|^{2\alpha}\mathcal{F}(\omega)(\xi), \quad \forall\ \alpha \in (0,1),$$

where $\mathcal{F}$ is the Fourier transform, i.e.,

$$\mathcal{F}(\omega)(\xi) = \frac{1}{(2\pi)^{\frac{N}{2}}}\int_{\mathbb{R}^N} exp\{-2\pi i\xi \cdot x\}dx.$$

If ω is smooth enough, $(-\Delta)^\alpha$ can be computed by

$$(-\Delta)^\alpha \omega(x) = c_{N,\alpha}P.V.\int_{\mathbb{R}^N} \frac{w(x)-w(y)}{|x-y|^{2\alpha+N}}dy,$$

where $P.V.$ is the principal value and $c_{N,\alpha}$ is a normalization constant which depends on N and α, precisely given by $c_{N,\alpha} = \left(\int_{\mathbb{R}^N} \frac{1-cos\xi_1}{|\xi|^{N+2\alpha}}d\xi\right)^{-1}$.

A basic motivation of the study of system (1.1) comes from standing wave solutions with the type of $\psi(x,t) = exp(-ict)u(x)$ for the time-dependent fractional Schröodinger equation

$$i\frac{\partial\psi}{\partial t} + (-\triangle)^\alpha\psi + (V(x)-c)\psi = f(x,\psi) + h(x),\ \ in\ \mathbb{R}^N,$$

where i is the imaginary unit.

When $\alpha = 1$, (1.1) becomes the classical Schrödinger-Poisson equation

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u), & x \in \mathbb{R}^N, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^N. \end{cases} \tag{1.2}$$

In the past decade, the existence and multiplicity of standing wave solutions of (1.2) have been widely investigated, we refer the readers to [1], [2], [3], [11] and the references therein. For fractional Schrödinger equations

$$(-\Delta)^\alpha u + V(x)u = f(x,u), \quad x \in \Omega, \tag{1.3}$$

where $\alpha \in (0,1)$ and Ω is a domain of $\mathbb{R}^N$. When Ω is a bounded domain, Nyamoradi [15] studied a class of Kirchhoff nonlocal fractional equation and obtained three solutions by using three critical point theorem. [12] studied a class of nonlocal fractional Laplacian equations depending on two real parameters and obtained the existence of three weak solutions. For more related results, we refer the readers to [13], [14] and the references therein.

When $V(x) = 1$, Dipierro et al. [4] proved the existence and symmetry of the solutions with $f(x,u) = |u|^{p-2}u$, and in [10], Felmer et al. obtained the existence, regularity and qualitative properties of ground states of problem (1.3), and the result has been extended in any dimension when α is sufficiently close to 1 by Fall et al. [5].

In particular, let $V(x) \equiv 1$ and $f(x,u) = |u|^{p-2}u$, the problem (1.1) will be reduced to

$$\begin{cases} (-\Delta)^\alpha u + u + \phi u = |u|^{p-2}u + h(x),\ x \in \mathbb{R}^N, \\ (-\Delta)^\alpha\phi = K_\alpha u^2,\ x \in \mathbb{R}^N. \end{cases} \tag{1.4}$$

In this paper, we mainly consider the problem (1.1) in $\mathbb{R}^N$. By combining the Mountain Pass theorem and the Ekeland variational principle, we obtain the existence of the two nontrivial solutions for problem (1.1). As a special case, we prove the existence of two nontrivial radial solutions for the problem (1.4).

II. PRELIMINARIES

In order to obtain multiple solutions for problem (1.1), we assume that $V(x)$ and $f(x,u)$ satisfy the following hypotheses

(v_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $\inf\limits_{x\in\mathbb{R}^N} V(x) \geq a_1 > 0$, where $a_1 > 0$ is a constant. Moreover, for any $M > 0$, $meas\{x \in \mathbb{R}^N : V(x) \leq M\} < \infty$, where meas denotes the Lebesgue measure in $\mathbb{R}^N$.

(f_1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, and there exist $2 < p < 2^*_\alpha = \frac{2N}{N-2\alpha}$ and constant $c_1 > 0$ such that

$$|f(x,u)| \leq c_1(1+|u|^{p-1}), \quad \forall\ x \in \mathbb{R}^N,\ u \in \mathbb{R}.$$

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(f_2) $\frac{f(x,u)}{u} \to 0$, as $u \to 0$ uniformly for $x \in \mathbb{R}^N$.

(f_3) There exist $\mu > 4$ such that $\mu F(x,u) \le uf(x,u)$, where $F(x,u) = \int_0^u f(x,s)ds$.

(f_4) $\inf\limits_{x\in\mathbb{R}^N, ||u||=1} F(x,u) > 0$.

In view of the potential $V(x)$, we consider the space

$$E = \left\{ u \in H^\alpha(\mathbb{R}^N) \Big| \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u|^2 + V(x)u^2 dx < \infty \right\}.$$

E is a Hilbert space with the inner product

$$(u,v)_E = \int_{\mathbb{R}^N} \left(|\xi|^{2\alpha} \hat{u}(\xi)\hat{v}(\xi) + \hat{u}(\xi)\hat{v}(\xi) \right) d\xi$$

$$+ \int_{\mathbb{R}^N} V(x)u(x)v(x)dx, \quad \forall u,\ v \in E.$$

and the norm

$$||u||_E = \left(\int_{\mathbb{R}^N} \left(|\xi|^{2\alpha}\hat{u}^2 + \hat{u}^2 \right) d\xi + \int_{\mathbb{R}^N} V(x)u^2 dx \right)^{\frac{1}{2}}.$$

is equivalent to the following norm

$$||u|| = \left(\int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{\alpha}{2}} u|^2 + \int_{\mathbb{R}^N} V(x)u^2 \right) dx \right)^{\frac{1}{2}}.$$

The corresponding norm is

$$(u,v) = \int_{\mathbb{R}^N} \left((-\Delta)^{\frac{\alpha}{2}} u (-\Delta)^{\frac{\alpha}{2}} v + V(x)uvdx \right), \ \forall u,\ v \in E.$$

In this paper, we will use the norm $|| \cdot ||$ in E.

Lemma 2.1 *([7]) For $1 < p < \infty$ and $0 < \alpha < \frac{N}{p}$, we have*

$$\|u\|_{\frac{pN}{N-p\alpha}} \le B||(-\Delta)^{\frac{\alpha}{2}} u||_p \tag{2.1}$$

with best constant $B = 2^{-\alpha}\pi^{\frac{-\alpha}{2}} \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{N+\alpha}{2})} \left(\frac{\Gamma(N)}{\Gamma(\frac{N}{2})} \right)^{\frac{\alpha}{N}}$.

Lemma 2.2. *For every $u \in H^\alpha(\mathbb{R}^N)$ there exists a unique $\phi = \phi_u \in D^\alpha(\mathbb{R}^N)$ which solves the equation $(-\Delta)^\alpha \phi = K_\alpha u^2, \ \ x \in \mathbb{R}^N$. Furthermore, ϕ_u is given by*

$$\phi_u = \int_{\mathbb{R}^N} |x-y|^{2\alpha-3} u^2(y)dy. \tag{2.2}$$

The mapping $\Phi : u \in H^\alpha(\mathbb{R}^N) \mapsto \phi_u \in D^\alpha$ and

$$[\phi_u]'(v) = 2\int_{\mathbb{R}^N} |x-y|^{2\alpha-3} u(y)v(y)dy, \ \forall\, u,v \in H^\alpha(\mathbb{R}^N). \tag{2.3}$$

Lemma 2.3. *Assume that a sequence $\{u_n\} \subset E$, $u_n \rightharpoonup u$, in E, as $n \to \infty$ and $\{u_n\}$ be a bounded sequence. Then*

$$\left| \int_{\mathbb{R}^N} \left(\phi_{u_n} u_n - \phi_u u \right)(u_n - u)dx \right| \to 0, \quad as\ n \to \infty. \tag{2.4}$$

Throughout the paper, $L^r(\mathbb{R}^N) = \{u : \mathbb{R}^N \to \mathbb{R} :$ u is measurable and $\int_{\mathbb{R}^N} |u|^r \, \mathrm{d}x < \infty\}$ with the norm $\|u\|_r = \left(\int_{\mathbb{R}^N} |u|^r \, \mathrm{d}x \right)^{\frac{1}{r}}$ and C is various positive generic constants, which may vary from line to line. Also if we take a subsequence of a sequence $\{u_n\}$ we shall denote it as $\{u_n\}$ again.

III. Conclusion

System (1.1) is the Euler-Lagrange equations corresponding to the functional $J : H^\alpha(\mathbb{R}^N) \times D^\alpha(\mathbb{R}^N) \to \mathbb{R}$ $J(u,\phi)$

$$= \frac{1}{2}\int_{\mathbb{R}^N} \Big(|(-\Delta)^{\frac{\alpha}{2}} u|^2 + \int_{\mathbb{R}^N} V(x)u^2 - \frac{1}{2}|(-\Delta)^{\frac{\alpha}{2}}\phi|^2$$

$$+ K_\alpha \phi u^2 \Big) dx - \int_{\mathbb{R}^N} F(x,u)dx - \int_{\mathbb{R}^N} h(x)udx. \tag{3.1}$$

By Lemma 1 in [8] and (2.2), the function J belongs to $C^1(H^\alpha(\mathbb{R}^N) \times D^\alpha(\mathbb{R}^N), \mathbb{R})$ and the partial derivatives in (u,ϕ) are given, for $\xi \in H^\alpha(\mathbb{R}^N)$ and $\eta \in D^\alpha(\mathbb{R}^N)$, we have

$$\left\langle \frac{\partial J}{\partial u}(u,\phi), \xi \right\rangle = \int_{\mathbb{R}^N} \Big((-\Delta)^{\frac{\alpha}{2}} u (-\Delta)^{\frac{\alpha}{2}} \xi + V(x)u\xi$$

$$+ K_\alpha \phi u \xi \Big) dx - \int_{\mathbb{R}^N} f(x,u)\xi dx - \int_{\mathbb{R}^N} h(x)\xi dx,$$

$$\left\langle \frac{\partial J}{\partial \phi}(u,\phi), \eta \right\rangle = \frac{1}{2}\int_{\mathbb{R}^N} \left(-(-\Delta)^{\frac{\alpha}{2}} \phi (-\Delta)^{\frac{\alpha}{2}} \eta + K_\alpha u^2 \eta \right) dx.$$

Thus, we have the following result:

Proposition 3.1. *The pair (u,ϕ) is a weak solution of system (1.1) if and only if it is a critical point of J in $H^\alpha(\mathbb{R}^N) \times D^\alpha(\mathbb{R}^N)$.*

We can consider the functional $J : H^\alpha(\mathbb{R}^N) \to \mathbb{R}$ defined by $J(u) = J(u,\phi_u)$. After multiplying equation $(-\Delta)^\alpha \phi_u = K_\alpha u^2$ by ϕ_u and integration by parts over $\mathbb{R}^N$, we have

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} \phi_u|^2 dx = K_\alpha \int_{\mathbb{R}^N} \phi_u u^2 dx. \tag{3.2}$$

By (3.2), the reduced functional takes the form

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{\alpha}{2}} u|^2 + \int_{\mathbb{R}^N} V(x)u^2 \right) dx$$

$$+ \frac{1}{4}K_\alpha \int_{\mathbb{R}^N} \phi_u u^2 dx - \int_{\mathbb{R}^N} F(x,u)dx - \int_{\mathbb{R}^N} h(x)udx. \tag{3.3}$$

Evidently, J is well defined and belongs to $C^1(E,\mathbb{R})$ with the derivative given by

$$\langle J'(u), v \rangle = \int_{\mathbb{R}^N} (-\Delta)^{\frac{\alpha}{2}} u (-\Delta)^{\frac{\alpha}{2}} v dx + \int_{\mathbb{R}^N} V(x)uvdx$$

$$+ K_\alpha \int_{\mathbb{R}^N} \phi_u uvdx - \int_{\mathbb{R}^N} f(x,u)vdx - \int_{\mathbb{R}^N} h(x)vdx. \tag{3.4}$$

It can be proved that $(u,\phi) \in E \times D^\alpha(\mathbb{R}^N)$ is a solution of system (1.1) if and only if $u \in E$ is a critical point of the functional J and $\phi = \phi_u$.

We consider a minimization of J constrained in a neighborhood of zero by using the Ekeland's variational principle, a critical point of J which achieves the local minimum of J can be found and the level of this local minimum is negative. Around the "zero" point, due to the Mountain Pass Theorem, we can also obtain a critical point of J and its level is positive. Therefore, the two critical points must be distinct, since they are in different levels.

Lemma 3.1. *Suppose that* (V_1), (f_1) *and* (f_2) *hold, let* $h \in L^2(\mathbb{R}^N)$ *hold. Then there exist some constants* $\rho,\ \eta,\ m_0 > 0$ *such that* $inf\{J(u) : u \in E$ *with* $||u|| = \rho\} \geq \eta$ *for all* $||h||_2 < m_0$.

Lemma 3.2. *Suppose that* (V_1), (f_3) *and* (f_4) *hold, then there exists a function* $v \in E$ *with* $||v|| > \rho$ *such that* $J(v) < 0$, *where* ρ *is given in Lemma 3.1.*

Lemma 3.3. *Assume that* (V_1), (f_1), (f_2) *hold and* $\{u_n\} \subset E$ *is a bounded Palais-Smale sequence of J, then* $\{u_n\}$ *has a strongly convergent subsequence in* E.

In view of the above discussions, we show the existence of at least two different solutions of the system (1.1).

Theorem 1. *Suppose that* $h \in L^2(\mathbb{R}^N)$ *and* $h \not\equiv 0$. *Let* (v_1) *and* $(f_1) - (f_4)$ *hold, then there exists a constant* $m_0 > 0$ *such that the system (1.1) possesses at least two different solutions when* $\|h\|_2 < m_0$.

Proof. The proof of this theorem is divided into two steps.

Step 1. There exists a function $u_1 \in E$ such that $J'(u_1) = 0$ and $J(u_1) < 0$.

For any $(x, u) \in \mathbb{R}^N \times \mathbb{R}$, set $h(t) = F(x, t^{-1}u)t^\mu$, $\forall\ t \in [1, +\infty)$. By (f_3), we have

$$\begin{aligned} h'(t) &= f(x, t^{-1}u)(-\frac{u}{t^2})t^\mu + F(x, t^{-1}u)\mu t^{\mu-1} \\ &= t^{\mu-1}\big[\mu F(x, t^{-1}u) - t^{-1}uf(x, t^{-1}u)\big] \leq 0 \end{aligned} \tag{3.5}$$

which implies that $h(t)$ is nonincreasing. Therefore, for any $|u| \geq 1$, we have $h(1) \geq h(|u|)$, that is,

$$F(x, u) \geq F(x, |u|^{-1}u)|u|^\mu \geq c_2|u|^\mu, \tag{3.6}$$

where $c_2 = \inf_{x\in\mathbb{R}^N, ||u||=1} F(x, u) > 0$ by (f_4). It follows from (f_2) that there exists $\alpha > 0$ such that

$$\Big|\frac{f(x, u)u}{u^2}\Big| = \Big|\frac{f(x, u)}{u}\Big| \leq 1, \tag{3.7}$$

for all $x \in \mathbb{R}^N$ and $0 < |u| \leq \alpha$. By (f_1), for $a.e.\ x \in \mathbb{R}^N$ and $\alpha \leq |u| \leq 1$, there exists $M_1 > 0$ satisfying

$$\Big|\frac{f(x, u)u}{u^2}\Big| \leq \frac{c_1(|u| + |u|^{p-1}|u|)}{u^2} \leq M_1. \tag{3.8}$$

Now, from (3.7), (3.8) we have that, for all for $a.e.\ x \in \mathbb{R}^N$ and $0 \leq |u| \leq 1$

$$f(x, u)u \geq -(M_1 + 1)|u|^2.$$

Using the inequality $F(x, u) = \int_0^u f(x, s)ds$, we have

$$F(x, u) \geq -\frac{1}{2}(M_1 + 1)|u|^2, \tag{3.9}$$

for $a.e.\ x \in \mathbb{R}^N$ and $0 \leq |u| \leq 1$. Taking $c_3 = \frac{1}{2}(M_1 + 1) + c_2$, then it follows from (3.6) and (3.9) that

$$F(x, u) \geq c_2|u|^\mu - c_3|u|^2, \tag{3.10}$$

for $a.e.\ x \in \mathbb{R}^N$ and $u \in \mathbb{R}$. Since $h \in L^2(\mathbb{R}^N)$ and $h \not\equiv 0$, we can choose a function $\psi \in E$ such that $\int_{\mathbb{R}^N} h(x)\psi > 0$. Therefore, we obtain that

$$\begin{aligned} J(t\psi) &= \frac{t^2}{2}\int_{\mathbb{R}^N}\big(|(-\Delta)^{\frac{\alpha}{2}}\psi|^2 + \int_{\mathbb{R}^N} V(x)\psi^2\big) \\ &+\frac{t^4}{4}K_\alpha\int_{\mathbb{R}^N}\phi_\psi\psi^2dx - \int_{\mathbb{R}^N} F(x, t\psi)dx - t\int_{\mathbb{R}^N} h(x)\psi dx \\ &\leq \frac{1}{2}t^2||\psi||^2 + Ct^4||\psi||^4 - c_2t^\mu||\psi||_\mu^\mu + c_3t^2||\psi||_2^2 \\ &- t\int_{\mathbb{R}^N} h(x)\psi dx < 0, \end{aligned}$$

for $t > 0$ small enough. Hence, we have

$$c_1 = inf\{J(u) : u \in \bar{B}_\rho\} < 0,$$

where $\rho > 0$ is given by Lemma 3.1. Ekeland's variational principle, there exists a sequence $\{u_n\} \subset \bar{B}_\rho$ such that $c_1 \leq J(u_n) < c_1 + \frac{1}{n}$, and $J(w) \geq J(u_n) - \frac{1}{n}||w - u_n||$ for all $w \in \bar{B}_\rho$. Applying a standard procedure, we can show that $\{u_n\}$ is a bounded $(PS)_c$ sequence of J. Hence, there exists a function $u_1 \in E$ such that $J'(u_1) = 0$ and $J(u_1) < 0$.

Step 2. There exists a function $\widetilde{u}_1 \in E$ such that $J'(\widetilde{u}_1) = 0$ and $J(\widetilde{u}_1) > 0$. By Lemma 3.1, 3.2 and Mountain Pass Theorem, there exists a sequence $\{u_n\} \subset E$ such that $J(u_n) \to \widetilde{c}_1 > 0, and J'(u_n) \to 0,\ \ as\ n \to \infty$. It follows from (f_3) that

$$\begin{aligned} &c_2 + 1 + ||u_n|| = J(u_n) - \tfrac{1}{\mu}\langle J'(u_n), u_n\rangle \\ &= (\frac{1}{2}-\frac{1}{\mu})||u_n||^2 + (\frac{1}{4}-\frac{1}{\mu})K_\alpha\int_{\mathbb{R}^N}\phi_{u_n}u_n^2dx \\ &+\int_{\mathbb{R}^N}(\frac{1}{\mu}f(x, u_n)u_n - F(x, u_n))dx + (\frac{1}{\mu}-1)\int_{\mathbb{R}^N} h(x)u_ndx \\ &\geq (\frac{1}{2}-\frac{1}{\mu})||u_n||^2 + \int_{\mathbb{R}^N}(\frac{1}{\mu}f(x, u_n)u_n - F(x, u_n))dx \\ &+(\frac{1}{\mu}-1)\int_{\mathbb{R}^N} h(x)u_ndx, \end{aligned}$$

for n large enough. Therefore, $\{u_n\}$ is bounded in E, since $\mu > 4$ and $||h||_2 < m_0$. □

Obviously, the system (1.4) has a variational structure. Indeed we consider the functional $I : H^\alpha(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}^N}\big(|(-\Delta)^{\frac{\alpha}{2}}u|^2 + u^2\big)dx + \frac{1}{4}K_\alpha\int_{\mathbb{R}^N}\phi_u u^2dx \\ &- \frac{1}{p}\int_{\mathbb{R}^N}|u|^pdx - \int_{\mathbb{R}^N} h(x)udx. \end{aligned} \tag{3.11}$$

Evidently, I is well defined and belongs to $C^1(H^\alpha(\mathbb{R}^N), \mathbb{R})$ with the derivative given by

$$\begin{aligned} \big\langle I'(u), v\big\rangle &= \int_{\mathbb{R}^N}(-\Delta)^{\frac{\alpha}{2}}u(-\Delta)^{\frac{\alpha}{2}}vdx + \int_{\mathbb{R}^N} uvdx \\ &+K_\alpha\int_{\mathbb{R}^N}\phi_u uvdx - \int_{\mathbb{R}^N}|u|^{p-2}uvdx - \int_{\mathbb{R}^N} h(x)vdx. \end{aligned} \tag{3.12}$$

Since system (1.4) is set on $\mathbb{R}^N$, it is well known that the Sobolev embedding $H^\alpha(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ is not compact for $2 \leq r < 2^*_\alpha$, then it is not easy to prove a minimizing sequence or a Palais-Smale sequence is strongly convergent if we look for solutions of system (1.4) via variational methods. In order to overcome the difficulty, we restrict problem (1.4) in the radial function space, where functions $u = u(r),\ \ r = |x|$. More precisely we shall consider I on the space of radial functions

$$H^\alpha_r(\mathbb{R}^N) := \{u \in H^\alpha(\mathbb{R}^N) : u = u(r),\ \ r = |x|\}.$$

$H_r^\alpha(\mathbb{R}^N)$ is a natural constraint for I, that is, any critical points $u \in H_r^\alpha(\mathbb{R}^N)$ of $I\big|_{H_r^\alpha(\mathbb{R}^N)}$ is also a critical point of I. Therefore, It's reduced to seek critical points of $I\big|_{H_r^\alpha(\mathbb{R}^N)}$.

Lemma 3.4. *Assume that $h \in C^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ is a radial function. Then there exist some constants $\rho_1,\ \eta_1,\ m_1 > 0$ such that $inf\{I(u) : u \in H^\alpha(\mathbb{R}^N)$ with $||u||_{H^\alpha} = \rho_1\} \geq \eta_1$ for all $||h||_2 < m_1$.*

Lemma 3.5. *There exists a function $v_1 \in H_r^\alpha(\mathbb{R}^N)$ with $||v_1||_{H^\alpha} > \rho_1$ such that $I(v_1) < 0$, where ρ_1 is given in Lemma 3.4.*

Lemma 3.6. *Assume that $\{u_n\} \subset H_r^\alpha(\mathbb{R}^N)$ is a bounded Palais-Smale sequence of I, then $\{u_n\}$ has a strongly convergent subsequence in $H_r^\alpha(\mathbb{R}^N)$.*

Theorem 2. *Suppose that $h \in C^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ is a radial function and $h \not\equiv 0$. Let $4 < p < 2_\alpha^*$, then there exists a constant $m_1 > 0$ such that problem (1.4) possesses at least two different solutions when $||h||_2 < m_1$.*

Proof. Step 1, we try to seek a function $u_2 \in H_r^\alpha(\mathbb{R}^N)$ such that $I'(u_2) = 0$ and $I(u_2) < 0$.

Since $h \in C^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ is a radial function and $h \not\equiv 0$, we can choose a function $\psi_1 \in H_r^\alpha(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} h(x)\psi_1 > 0.$$

Therefore, we obtain that

$$I(t\psi_1) = \frac{t^2}{2}\int_{\mathbb{R}^N}(|(-\Delta)^{\frac{\alpha}{2}}\psi_1|^2 + \int_{\mathbb{R}^N}\psi_1^2)dx$$

$$+\frac{t^4}{4}K_\alpha\int_{\mathbb{R}^N}\phi_{\psi_1}\psi_1^2dx - \frac{t^p}{p}\int_{\mathbb{R}^N}|\psi_1|^pdx - t\int_{\mathbb{R}^N}h(x)\psi_1dx$$

$$\leq \frac{t^2}{2}||\psi_1||^2 + Ct^4||\psi_1||^4 - \frac{t^p}{p}||\psi_1||_p^p - t\int_{\mathbb{R}^N}h(x)\psi_1dx < 0,$$

for $t > 0$ small enough. Hence, we have $c_2 = inf\{I(u) : u \in \bar{B}_{\rho_1}\} < 0$, where $\rho_1 > 0$ is given by Lemma 3.4. Then by Ekeland's variational principle, there exists a sequence $\{u_n\} \subset \bar{B}_{\rho_1}$ such that

$$c_2 \leq I(u_n) < c_2 + \frac{1}{n}$$

and

$$I(w) \geq I(u_n) - \frac{1}{n}||w - u_n||_{H^\alpha}$$

for all $w \in \bar{B}_{\rho_1}$. It follows from Lemma 3.6 that there exists a function $u_2 \in H_r^\alpha(\mathbb{R}^N)$ such that $I'(u_2) = 0$ and $I(u_2) < 0$.

Step 2. There exists a function $\widetilde{u}_2 \in H_r^\alpha(\mathbb{R}^N)$ such that $I'(\widetilde{u}_2) = 0$ and $I(\widetilde{u}_2) > 0$. By Lemma 3.4, 3.5 and Mountain Pass Theorem, there exists a sequence $\{u_n\} \subset H_r^\alpha(\mathbb{R}^N)$ such that $I(u_n) \to \widetilde{c}_2 > 0, I'(u_n) \to 0, as\ n \to \infty$. From Lemma 3.6, we only need to check that $\{u_n\}$ is bounded in $H_r^\alpha(\mathbb{R}^N)$. By (3.12), $4 < p < 2_\alpha^*$, we have

$$p\tilde{c}_2 + ||u_n||_{H^\alpha} = pI(u_n) - \langle I'(u_n), u_n\rangle$$

$$= (\frac{p}{2} - 1)||u_n||_{H^\alpha}^2 + \frac{(p-4)K_\alpha}{4}\int_{\mathbb{R}^N}\phi_{u_n}u_n^2dx$$

$$-(p-1)\int_{\mathbb{R}^N}h(x)u_ndx$$

$$\geq (\frac{p}{2} - 1)||u_n||_{H^\alpha}^2 - (p-1)||h||_2||u_n||_{H^\alpha},$$

for n large enough. Therefore, $\{u_n\}$ is bounded in $H_r^\alpha(\mathbb{R}^N)$. □

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References

[1] A. Azzollini, A. Pomponio, *Ground state solutions for the nonlinear Schrödinger-Maxwell equations*, J. Math. Anal. Appl., 345 (2008), 90-108.

[2] Wang L X, Ma Sh W, *Multiple solutions for nonhomogeneous Schrödinger-Poisson equations with sign-changing potential*, Acta Mathematica Sciential., 37B (2017), 555-572.

[3] Sun J T, Wu T F, *Existence and multiplicity of positive solutions for a Schrodinger-Poisson system with a perturbation*,Topological Methods in Nonlinear Analysis, 46 (2015), 967-998.

[4] S. Dipierro, G. Palatucci, E. Valdinoci, *Existence and symmetry results for a Schrödinger type problem involving the fractional Laplacian*, Matematiche., 68 (2013), 201-216.

[5] M. Fall, E. Valdinoci, *Uniqueness and nondegeneracy of positive solutions of $(-\Delta)^s u + u = u^p$ in $\mathbb{R}^N$ when s is close to 1*, Commun. Math. Phys., 329 (2014), 383-404.

[6] R. Frank, E. Lenzmann, *Uniqueness and nondegeneracy of ground states for $(-\Delta)^s Q + Q - Q^{\alpha+1}$ in $\mathbb{R}$*, Acta. Math., 201 (2013), 261-318.

[7] H. Hajaiej, X. Yu, Z. Zhai, *Fractional Gagliardo-Nirenberg and Hardy inequalities under Lorentz norms*, J. Math. Anal. Appl., 396 (2012), 569-577.

[8] J. Jin, X. Wu, *Infinitely many radial solutions and Kirchhoff-type equations in $\mathbb{R}^N$*, J. Math. Anal. Appl., 369 (2010), 564-574.

[9] N. Laskin, *Fractional quantum mechanics and Lvy path integrals*, Phys Lett A., 268 (2000), 298-305.

[10] N. Laskin, *Fractional Schrödinger equation*, Phys. Rev. E., 66 (2002), 056108.

[11] H. Liu, H. Chen, *Multiple solutions for superlinear Schrödinger-Poisson system with sign-changing potential and nonlinearity*, Comput. Math. Appl., 68 (2014), 1982-1990.

[12] B.G. Molica, Repovš D., *Higher nonlocal problems with bounded potential*, J. Math. Anal. Appl., 420 (2014), 167-176.

[13] B.G. Molica, *Sequences of weak solutions for fractional equations*, Math. Res. Lett. 21(2) (2014)2, 241-253.

[14] B.G. Molica, B. Pansera, *Three weak solutions for nonlocal fractional equations*, Adv. Nonlinear Stud., 14(3) (2014), 619-629.

[15] N. Nyamoradi, *Existence of three solutions for Kirchhoff nonlocal operators of elliptic type*, Math. Commun., 18 (2013), 489-502.