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A Weighted Strike-Related Implied Volatility Model

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***Abstract*—The implied volatility in the Black-Scholes model is assumed as a constant. However, empirical analysis proves that the values of the implied volatility based on the same underlying asset vary with the underlying asset price, maturity date, time to maturity, and the strike price. Ronald Lagnado and Stanley Osher and Chiarella proposed the techniques for calibrating derivative pricing by solving an inverse problem. In this paper, an improved model is proposed. In which, the influence of the options with different strike prices is distinguished by weights and the choices of the weight functions are discussed. The approach we take is numerically solving an inverse problem which is more solid in theory than simple regression methods. Numerical results show that our model has better adaptivity and accuracy than traditional models.**

Keywords-implied volatility; the inverse problem method; B-S model; Poisson equation; options

I. INTRODUCTION

The option, as one of the most important financial derivatives, its pricing problem has become a hot issue among the fields of financial engineering. The Black-Scholes (B-S) model [3] supposes that the implied volatility is a constant. While both the empirical analysis and researches [6]show that the implied volatility obtained from the B-S model with the market data, vary with the underlying asset price, maturity date, time to maturity, and the strike price. That means the implied volatility is a function of these variables instead of a constant, which shows a curve with smile skew [10] [11]. Therefore, it is a key problem how to model implied volatility function [7] for accurate option pricing.

At present, the methods of modelling implied volatility include the methods based on statistical regression and the methods based on the inverse problem. The statistical regression methods construct the implied volatility function by using regression methods to approximate the statistical patterns between the implied volatility with strike price and the time to maturity, which usually being partitioned as the parametric models [5], semi-parametric models [2] [11] and non-parametric models.

The method of modelling implied volatility based on the inverse problem is to estimate the implied volatility function by solving the reverse problem associated with the B-S model and the observed market data. The method was first proposed by Ronald Lagnado and Stanley Osher in 1997 [9] in which they only considered initial prices of options. In 2000, Chiarella et al. [4] extended the model by using the option prices in a period to replace initial prices of options.

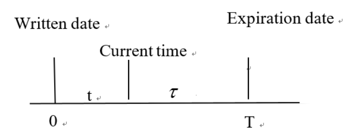

Figure 1. Maturity and time to maturity

In this paper, we try to construct an implied volatility model with fitting and forecasting ability improving the Osher model. The different computing functions are used to describe the implied volatilities related to the options with different strike prices and the smile skew of the implied volatility function may be explored. The difference among the computing functions lies in the weights, which is related to the strike prices and used to distinguish the influence of option prices with different strike prices. The weights are computed according to the difference among the strike prices. By using the Euler-Lagrange method, the model is converted to a Poisson Equation about implied volatilities and solved by using the iterative method. The experimental results show that the proposed model has higher accuracy compared with the traditional models.

II. RELATED WORKS

A. Options and Black-Scholes Pricing model

An option O is a contract writing on an underlying asset A that gives the owner of the option the right, but not the obligation, to buy or sell A at a specified strike price K on a specified date D (expiration). The action buying or selling depends on the form of the option, the former is called a call and the latter is a put. The maturity T is the time from the day the option is written to its expiration day. The current time t between 0 and T can be considered as the maturity degree and $\tau = T - t$ is the time-to-maturity. The relationship among t,T and τ is depicted in Fig.1.

At the current time t , suppose the prices of the option and the underlying asset are V and S respectively, then V is a function related to S and t :

$$V = V(S, t). \tag{1}$$

Under some ideal assumptions, the Black-Scholes model [3] illustrated that V and S satisfy the following

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equation:

$$\frac{\partial V}{\partial t}+rS\frac{\partial V}{\partial S}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}-rV=0, \tag{2}$$

where parameter r is the risk-free interest, parameter σ is the implied volatility. These two parameters are supposed as constants.

For simplicity, the call option is considered. The boundary conditions of the above equation are as follows:

$$\begin{cases} V(S,T)=max(S-K,0), & \text{for } S\geq 0,\\ V(0,T)=0, & \text{for } 0\leq t\leq T,\\ \frac{\partial V}{\partial S}(S,t)\rightarrow e^{-q(T-t)}, \text{as } S\rightarrow\infty, & \text{for } 0\leq t\leq T. \end{cases} \tag{3}$$

Solving the (2) under the boundary conditions (3), the following formula can be obtained for computing the price of a call option:

$$\begin{cases} V(S,T)=S\times N(d_1)-Ke^{-r(T-t)}\times N(d_2)\\ d_1=\frac{ln\frac{S}{K}+(r+\frac{\sigma_2}{2})(T-t)}{\sigma\sqrt{T-t}}\\ d_2=\frac{ln\frac{S}{K}-(r+\frac{\sigma_2}{2})(T-t)}{\sigma\sqrt{T-t}} \end{cases}. \tag{4}$$

where $N(\cdot)$ is the standard normal density distribution function.

From above equations, one may find that K,T and σ are the parameters of the option price. So, the formula (1) can be rewritten as the following:

$$V=V(S,t;K,T,\sigma). \tag{5}$$

B. Implied Volatility Model Based on the Inverse Problem

Due to the volatility smiles and/or term structures explored by a lot of empirical analysis, Lagnado and Osher [9] proposed that the implied volatility of the B-S model is a function of underlying price S and the current time t rather than a constant.Hence the B-S differential equation (1) can be rewritten as following:

$$\frac{\partial V}{\partial t}+rS\frac{\partial V}{\partial S}+\frac{1}{2}\sigma^2(S,t)S^2\frac{\partial^2 V}{\partial S^2}-rV=0. \tag{6}$$

Suppose the set Φ_A includes all the options writing on the same underlying asset in the market. The maturities of the options in Φ_A include $T_i, 1\leq i\leq N$ and the strike prices may be $K_j, j=1,\ldots M$. Denote O_{ij} be the option with maturity T_i , the strike price K_j . At the maturity degree t , the price of the asset A is S and the option price $V_{ij}(S,t)$ (if there is not such option in the market, the price of the option will be set with 0 ; and if there are more than one such options, the price of the option will be set the average) . Then the options set can be defined as follows:

$$\Phi_A=\{O_{ij}|1\leq i\leq N, 1\leq j\leq M\}. \tag{7}$$

In the other hand, if the implied volatility function $\sigma(S,t)$ is given, the theoretical price $V(S,t;K,T,\sigma)$ of the option O_{ij} can be computed by solving the (6) under the condition (3). A good $\sigma(S,t)$ will make the error between the theoretical option price and the market data, $|V(S,t;K,T,\sigma)-V_{ij}(S,t)|$, small.

From this point of view, Lagnado and Osher proposed an energy functional. They considered the initial situation of O_{ij} . At this moment, $t=0, S=S_0$. There is no market price about O_{ij} but the ask and bid prices V_{ij}^a and V_{ij}^b . So the average of V_{ij}^a and V_{ij}^b is used to replace $V_{ij}(S_0,0)$. After these, they minimized the following energy functional to obtain the best implied volatility function:

$$\begin{cases} F(\sigma)=\|\nabla\sigma\|_2^2+\lambda\sum_{i=1}^{N}\sum_{j=1}^{M}[V(S_0,0;K_j,T_i,\sigma)-\bar{V_{ij}}]^2\\ \bar{V_{ij}}=\frac{1}{2}(V_{ij}^a+V_{ij}^b). \end{cases} \tag{8}$$

where the term $\|\nabla\sigma\|_2^2$ is for regularization and $\lambda>0$ is a constant.

Chiarella et al. [4] extended the above model by using the information during a period to replace the initial information and proposed the following energy functional:

$$H(\sigma)=\|\nabla\sigma\|_2^2+ \lambda\sum_{i=1}^{N}\sum_{j=1}^{M}\int_0^\infty\int_0^{T_{cur}}[V(S,t;K_j,T_i,\sigma)-\bar{V_{ij}}]^2dtdS. \tag{9}$$

where T_{cur} is the current time.

III. A Strike-related Implied Volatility Model

A. The Energy Functional of the New Model

The above two models assumed that $\sigma=\sigma(S,t)$, that means, the term structure explored by the empirical analysis has been considered which is related to the time-to-maturity ($\tau=T-t$). But the other phenomena, implied volatility smile, has not been considered yet.

In this paper, the implied volatility of an option is assumed related to the current time t , the price of the underlying asset $S=S(t)$ at t , the strike price K and the maturity T . That is,

$$\sigma=\sigma(S,t,K,T). \tag{10}$$

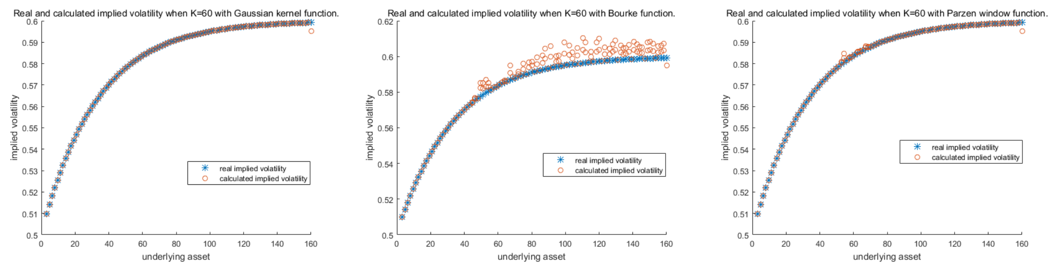
Fixing the maturity, for a given strike price K, there is a corresponding implied volatility function $\sigma_K=\sigma_K(S,t)$. Using the same symbols with Section II-B, a new energy functional is proposed as following:

$$L_\lambda(\sigma_K)=\|\nabla\sigma\|_2^2+\lambda\sum_{i=1}^{N}\sum_{j=1}^{M}\{\alpha_K(j)\cdot \int_0^\infty\int_0^{T_{cur}}[V(S,t;K_j,T_i,\sigma)-\bar{V_{ij}}]^2dtdS\}. \tag{11}$$

where $\alpha_K(j)$ is the weight used for exploring the influence of the options with the strike price K_j . The closer K_j to K , the more influence of the price of the option with the strike price K_j , and vice versa.

Similar to the [4], the minimization of the functional (11) can be converted to solve the following Euler-Lagrange equation by using the variational method:

$$\frac{\partial^2\sigma_K}{\partial S^2}+\frac{\partial^2\sigma_K}{\partial t^2}=\lambda\cdot\sum_{i=1}^{N}\sum_{j=1}^{M}[\alpha_K(j)\cdot \frac{\partial V(S,t,K_j,T_i,\sigma_K)}{\partial\sigma_K}V(S,t,K_j,T_i,\sigma_K-V_{ij})]. \tag{12}$$



(a) The results by using Gaussian kernel function

(b) The results by using Bourke function

(c) The results by using Parzen window function

Figure 2. Simulated implied volatilities with $K = 60$ by using different weights

To solve (12), the boundary functions $\sigma_K(S,0)$, $\sigma_K(0,t)$, $\sigma_K(S_{max},t)$,$\sigma_K(S,t_{max})$ need to be given. If the price of the underlying asset is 0, the option writing on it is invalid. Based on this, it is assumed that $\sigma_K(0,t) = 0$. The other three boundary functions cannot be obtained by theory and need to be fitted by using the market data.

B. The Choice of Weights

Suppose σ_K is the implied volatility related to the options with the strike price K . The weight function $\alpha_K(j), 1 \leq j \leq M$ is used to explore the sampling influence of the options with the strike K_j to σ_K . Analysing the market data, it can be found that the option with the strike price K_j has a greater influence to σ_K when the distance from the strike price K_j to K is closer, and a bigger weight is needed. In the other hand, considering the stability of the (12), the weights must decrease when the number of the sampling data increases. Based on these consideration, $\alpha_K(j)$ can use following functions:

1) Parzen window function [8]:

$$\alpha_K(j) \begin{cases} 1, & K_j - h \leq K \leq K_j + h \\ 0, & \text{else} \end{cases} . \tag{13}$$

2) Gaussian kernel function:

$$\alpha_K(j) = \frac{\frac{1}{\sqrt{2\pi}} e^{\frac{-(K-K_j)^2}{2}}}{\sum\limits_{l=1}^{M} \frac{1}{\sqrt{2\pi}} e^{\frac{-(K-K_l)^2}{2}}}. \tag{14}$$

3) Bourke function [1]:

$$\alpha_K(j) = \frac{\frac{1}{(K_j-K)^2}}{\sum\limits_{l=1}^{M} \frac{1}{(K_l-K)^2}}. \tag{15}$$

C. Numerical Computing

An iterative method is used to solve (12) numerically in this paper. Given an initial implied volatility value, the theoretical prices of the options of the right- hand side term of the equation $V(S,t,K_j,T_i,\sigma_K)$ can be computed by solving the Black-Scholes Equation where $S(t), K_j, T_i$ are the market data at the current time t . Then, this value is used to solve (12) and obtain a new implied volatility value. This new implied volatility value then is used to replace the given initial value and the above computing steps are repeated until a good implied volatility value obtained.

In order to numerical implementation, it is necessary to be discretized along S and t directions. The market data may not be in the grids. In this case, simple interpolations can be used to obtain necessary data.

IV. Numerical Simulation

To test the feasibility of the algorithm, the real implied volatility function is usually assumed to be known as σ_{real}. Using it, the corresponding real option price V_{ij} (simulating market data) can be computed by solving the B-S model.

During the validation, the implied volatility values obtained by solving the proposed model is compared with σ_{real} to analyze the effectiveness of the model.

In the following tests, similar to the [4], the real implied volatility function σ_{real} is set as $\sigma_{real} = 0.6(1 - e^{-0.03(S+K)})$. The other parameters are set as following: the underlying price $1.6 \leq S \leq 160$, and the step size of S is 1.6; the maturity is fixed as $T = 10/252$,$0 \leq t \leq 10/252$, and the step size of t is 1/252; the strike price $50 \leq K \leq 160$ and the step size of K is 10; the risk-free rate is 0.05; the parameter $\lambda = 1$.

A. Comparison of simulation errors with different weight functions

Three weight functions are given in Section III-B. The average simulation errors $|\sigma - \sigma_{real}|$ by using different weight functions are given in Table I where $h = 0$ for the Parzen window function. The simulation results of the implied volatilities with $K = 60$by using different weight functions are given respectively in Fig.2. One may find that the errors are small for all three weight functions. In the other hand, the Gaussian kernel function has a best result due to the continuity of σ_{real} which may not be appropriate in the real market situation.

B. Comparison of simulation errors with different models

Table II show the average simulation errors by using the model proposed in [4] and proposed model in this paper in which the weight function is chosen as the Parzen window function with $h = 0$. Fig.3 shows the simulation results by using the model of [4] and the proposed model in this paper. From the Table II, one can find that, compared with

Table I
SIMULATION ERRORS BY USING DIFFERENT WEIGHT FUNCTIONS

Strike price	Average errors		
	Gaussian kernel	Bourke	Parzen window
50	3.8115e-05	0.0016	0.000106
60	4.1069e-05	0.0032	0.000119
70	4.3268e-05	0.0050	0.000159
80	4.4901e-05	0.0068	0.000208
140	0.0013	0.0028	0.000255
150	0.0246	0.0253	0.024020

Table II
SIMULATION ERRORS BY USING DIFFERENT MODELS

Strike Price K	Average errors by using the model in [4]	Average errors of the proposed model
60	0.0017	0.000119
70	0.0022	0.000159
90	0.0030	0.000228
100	0.0032	0.000293
120	0.0028	0.000353
130	0.0023	0.000337

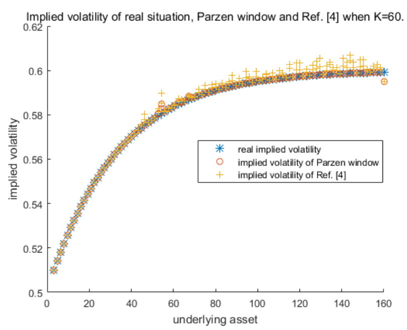


Figure 3. Implied volatility of real situation, Parzen window and [4] when $K = 60$

Table III
SIMULATION ERRORS IN DIFFERENT BOUNDARIES

Strike Price K	Average errors with linear function boundaries	Average errors with real boundaries
60	0.004121627	0.000119
70	0.004333177	0.000159
90	0.004605998	0.000228
100	0.004692007	0.000293
120	0.004802928	0.000353
130	0.004837897	0.000337

the methods proposed by the [4], our proposed model has a better precision.

C. Influence of different boundary conditions

The boundary conditions of the above simulation experiments use the real boundary function of the simulation implied volatility function σ_{real} . In fact, it is difficult to obtain the boundary functions of the market data because that the sampling data in the boundary is sparse. In this test, a liner fitting function of the boundary function of σ_{real} is used for as the boundary conditions which aims to find the influence of the boundary conditions. Table III shows the simulation errors in different boundaries.

One may find from the simulation experiments that, the errors are sensible with the boundaries. Good boundaries are necessary to have high accurate results.

V. CONCLUSIONS

In this paper, based on the inverse problem method, a strike-related implied volatility model is proposed. In the new model, the implied volatility is related to the option s strike price. And the options with different strike price have different influence on the underlying implied volatility. The simulation experimental results show the feasibility of the new model. Further search will focus on how to realize the model by real market data.

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